

Lowest Landau Level Stress Tensor and Structure Factor of Trial Quantum Hall Wave Functions

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We show that for a class of model Hamiltonians for which certain trial quantum Hall wavefunctions are exact ground states, there is a single spectral density function which controls all two-point correlation functions of density, current and stress tensor components. From this we show that the static structure factors of these wavefunctions behaves at long wavelengths as $s_4 k^4$ where the coefficient s_4 is directly related to the shift: $s_4 = (\mathcal{S} - 1)/8$.

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Introduction.—Starting from the work of Laughlin [1], trial wavefunctions have been playing a very important role in quantum Hall physics. Some of the most interesting quantum Hall phases have been predicted theoretically by the explicit construction of trial wavefunctions, most notably the Moore-Read (Pfaffian) state [2] and the Read-Rezayi parafermion series [3]. A more recent construction involves the Jack polynomials [4] and includes an earlier proposed “Gaffnian” state [5]; map of these states, however, are not expected to correspond to gapped quantum Hall states. In most cases, the trial wavefunction is a ground state of a model Hamiltonian containing only local interactions [6]. The study of these wavefunctions is important for the understanding of the properties of the quantum Hall phases.

In this paper we will show, among other results, that for a large class of trial wavefunctions, the low-wavelength asymptotics of the projected structure factor [7] is determined exactly by the shift,

$$s_4 \equiv \lim_{k \rightarrow 0} \frac{\bar{s}(k)}{(k\ell_B)^4} = \frac{\mathcal{S} - 1}{8}, \quad (1)$$

where ℓ_B is the magnetic length that we frequently will set to 1. This equation is already known to be valid for the Laughlin states with filling fractions $\nu = 1/(2n + 1)$ where $s_4 = (1 - \nu)/(8\nu)$ [7] and $\mathcal{S} = 1/\nu$. In this paper we show that Eq. (1) is valid also for the Moore-Read states, the Read-Rezayi parafermion states, both for bosons and fermions [23]. Equation (1) was speculated to be valid in Ref. [8] for all states whose wavefunctions are constructed from conformal field theory correlators. The statement has not been proven rigorously for any states beyond the Laughlin states. Instead the *inequality*,

$$\lim_{k \rightarrow 0} \frac{\bar{s}(k)}{k^4} \geq \frac{|\mathcal{S} - 1|}{8}, \quad (2)$$

has been shown to be valid for all gapped ground states on the lowest Landau level (LLL) [8, 9]. Equation (1) shows that the trial ground states are truly special in

their respective classes—these are the states that minimize the structure factor at small k .

Our proof of Eq. (1) is not a direct one, but but relies on some techniques which reveal some other unusual properties of the trial states. First we find the LLL expression for the components of the stress tensor. The LLL form of the electromagnetic current has been found in the past [10–12], but the stress tensor has not been obtained in these works. We then show explicitly that the trial ground states in the Read-Rezayi parafermion series (which includes the Moore-Read state) are annihilated by one component of the particle number current, as well as all but one components of the stress tensor. Namely,

$$J_{\bar{z}}(x)|0\rangle = 0, \quad T_{\bar{z}\bar{z}}(x)|0\rangle = T_{zz}(x)|0\rangle = 0, \quad (3)$$

where $|0\rangle$ denotes the ground state. These equations imply that all two-point correlation functions involving the density, the current, and the stress tensor are determined by one single spectral density. From this Eq. (1) follows.

Action principle for a system on the lowest Landau level.—In principle, the form of the stress tensor can be obtained by a procedure similar to the one followed in Refs. [10–12] for the electromagnetic current. One would develop a perturbation theory in the inverse cyclotron frequency and pick out the terms that survive when the cyclotron frequency goes to infinity. Here we use a much simpler method based on the Lagrangian formalism.

To derive the form of the stress tensor, we put the system in a curved metric. We consider the system of two-dimensional non-relativistic particles interacting with gauge potential A_μ in curved space [13] with $g = \det(g_{ij})$

$$S = \int d^3x \sqrt{g} \left[\frac{i}{2} (\psi^\dagger D_t \psi - D_t \psi^\dagger \psi) - \frac{g^{ij}}{2m} D_i \psi^\dagger D_j \psi + \frac{B}{2m} \psi^\dagger \psi + \mathcal{L}_{\text{int}} \right], \quad (4)$$

where ψ is the spinless field operator, \mathcal{L}_{int} the interacting Lagrangian density. To facilitate taking the $m \rightarrow 0$

limit we have added a magnetic moment term with the gyromagnetic ratio equal to 2; this term does not affect the ground state wavefunction in constant magnetic field. The magnetic field is $B = \frac{\epsilon^{ij}}{\sqrt{g}} \partial_i A_j$. The covariant derivatives are defined as

$$D_\mu = \partial_\mu - iA_\mu + i\omega_\mu, \quad (5)$$

where the spin connection in term of vielbein is [14]

$$\omega_t = \frac{1}{2} \epsilon_{ab} e^{aj} \partial_t e_j^b, \quad \omega_i = \frac{1}{2} \left(\epsilon_{ab} e^{aj} \partial_i e_j^b - \frac{\epsilon^{jk}}{\sqrt{g}} \partial_j g_{ik} \right). \quad (6)$$

The inclusion of ω_μ in the covariant derivative is optional, but it simplifies the Ward identities used later. For convenience, we define the complex vielbein vectors

$$e_i = \frac{1}{\sqrt{2}}(e_i^1 - ie_i^2), \quad \bar{e}_i = \frac{1}{\sqrt{2}}(e_i^1 + ie_i^2), \quad (7)$$

and introduce the following notation for the projection of each vector X_i on the complex vielbeins,

$$X = e^i X_i, \quad \bar{X} = \bar{e}^i X_i. \quad (8)$$

In flat space we can choose $e_i^a = \delta_i^a$, then $\partial = \sqrt{2}\partial_z$, $\bar{\partial} = \sqrt{2}\partial_{\bar{z}}$. For the lack of better terminology, we will call X the “holomorphic” component and \bar{X} the “antiholomorphic” component of X_i , without committing to any particular dependence of X and \bar{X} on the spatial coordinates.

The action (4) contains term that are singular in the $m \rightarrow 0$ limit. To have a smooth $m \rightarrow 0$ limit, we notice that $D_i D^i + B = D\bar{D}$ and use the Hubbard-Stratonovich transformation to rewrite the action in the form

$$S = \int d^3x \sqrt{g} \left[\frac{i}{2} (\psi^\dagger D_t \psi - D_t \psi^\dagger \psi) - D\psi^\dagger \chi - \chi^\dagger \bar{D}\psi + m\chi^\dagger \chi + \mathcal{L}_{\text{int}} \right]. \quad (9)$$

In the LLL limit $m \rightarrow 0$, χ and χ^\dagger play the role of Lagrange multipliers enforcing the constraint

$$\bar{D}\psi = 0, \quad D\psi^\dagger = 0. \quad (10)$$

This is simply the lowest Landau level constraint, which in flat space becomes $D_{\bar{z}}\psi = 0$. In the symmetric gauge $A_x = -\frac{1}{2}By$, $A_y = \frac{1}{2}Bx$ the constraint implies that ψ is proportional to a linear combination of $z^n e^{-B|z|^2/4}$. The time evolution follows the equation of motion

$$i\partial_t \psi + D\chi + A_t \psi + \frac{\delta \mathcal{L}_{\text{int}}}{\delta \psi^\dagger} = 0, \quad (11)$$

where $\mathcal{L}_{\text{int}} = \int d\vec{x} \mathcal{L}_{\text{int}}$, and its complex conjugate. In Eq. (11) the Lagrange multiplier χ is such that the constraint (10) is maintained at all times. We shall determine χ from this condition later.

We will mostly consider in this paper \mathcal{L}_{int} that contains only local interactions, i.e., interactions that are given by a product of ψ , ψ^\dagger and their derivatives at the same point. In the first-quantized language such an interaction corresponds to a many-body potential in the form of a product of delta functions and their derivatives. Upon projection onto the LLL, such interactions become the pseudopotential interactions. Due to the LLL constraint, any such potential can be written as $\mathcal{L}_{\text{int}}[\psi^\dagger, \psi, \bar{D}^n \psi^\dagger, D^n \psi]$.

The form of the interaction Lagrangian is chosen so that the ground state is the trial wave function under consideration. In this paper the interaction term that lead to a well-known series of trial wavefunctions: the Read-Rezayi parafermion series. For bosons, our interaction Lagrangian has the form

$$\mathcal{L}_{\text{int}} = -\lambda(\psi^\dagger)^k \psi^k, \quad (12)$$

and for fermions,

$$\mathcal{L}_{\text{int}} = -\lambda |\psi D\psi D^2\psi \dots D^{k-1}\psi|^2. \quad (13)$$

These interactions place an energy penalty on k particles coming in with minimal angular momentum. For case $k = 2$ corresponds to the $\nu = 1/2$ bosonic and $\nu = 1/3$ fermionic Laughlin states; $k = 3$ corresponds to the Moore-Read states, and $k > 3$ to the parafermion states.

Currents and stress tensor.—We define the charge current J^μ and the stress tensor T^{ij} from the variation of the action with respect to the background fields,

$$\delta S = \int d^3x \sqrt{g} \left(J^\mu \delta A_\mu + \frac{1}{2} T^{ij} \delta g_{ij} \right). \quad (14)$$

where the action is given by Eq. (9). For convenience, in addition to the holomorphic and antiholomorphic components of the current $J = J^i e_i$, $\bar{J} = J^{\bar{i}} \bar{e}_{\bar{i}}$, we introduce

$$T = T^{ij} e_i e_j = 2T_{zz}, \quad \tilde{T} = T^{i\bar{j}} \bar{e}_{\bar{i}} \bar{e}_{\bar{j}} = 2T_{\bar{z}\bar{z}}, \quad (15)$$

$$T^{\text{tr}} = T^{ij} e_i \bar{e}_{\bar{j}} = 2T_{z\bar{z}}. \quad (16)$$

We call T and \tilde{T} the holomorphic and antiholomorphic stress components; T^{tr} is simply the trace of the stress tensor. By varying the Lagrangian, we find

$$\rho \equiv J^0 = \psi^\dagger \psi, \quad J = i\chi^\dagger \psi, \quad \bar{J} = -i\psi^\dagger \chi. \quad (17)$$

Note that the \mathcal{L}_{int} , even in the fermion case, does not depend on the gauge potential A_μ and hence does not contribute to the current.

For the stress tensor we find

$$\tilde{T} = \bar{D}\psi^\dagger \chi - \bar{\partial}(\psi^\dagger \chi) + \bar{e}_i \frac{\delta \mathcal{L}_{\text{int}}}{\delta e_i}, \quad (18a)$$

$$T = \chi^\dagger D\psi - \partial(\chi^\dagger \psi) + e_i \frac{\delta \mathcal{L}_{\text{int}}}{\delta \bar{e}_i}, \quad (18b)$$

$$T^{\text{tr}} = 2\mathcal{L}_{\text{int}} + e_i \frac{\delta \mathcal{L}_{\text{int}}}{\delta e_i} + \bar{e}_i \frac{\delta \mathcal{L}_{\text{int}}}{\delta \bar{e}_i} - \frac{\delta \mathcal{L}_{\text{int}}}{\delta \psi} \psi - \psi^\dagger \frac{\delta \mathcal{L}_{\text{int}}}{\delta \psi^\dagger}. \quad (18c)$$

where χ is given in (26). Now the bosonic L_{int} does not contain any e^i and the last terms in Eqs. (18a) and (18b) equal zero. For the fermionic interaction (13), the operator $\bar{e}_i \delta / \delta e_i$ converts one D into a \bar{D} , which can be pushed to act on ψ by using $[\bar{D}, D] = -B$. Due to the constraints $\bar{D}\psi = 0$ the result will be a product of k derivatives of ψ but two of the derivatives will have the same power. Fermion statistics then implies that the result is zero. Thus we find that L_{int} does not contribute to the traceless components of the stress tensor and

$$\tilde{T} = \bar{D}\psi^\dagger \chi - \bar{\partial}(\psi^\dagger \chi), \quad (19a)$$

$$T = \chi^\dagger D\psi - \partial(\chi^\dagger \psi). \quad (19b)$$

Special properties of trial ground state.—We now show that the trial ground states satisfy the following properties:

$$\bar{J}(x)|0\rangle = 0, \quad (20)$$

$$\bar{T}(x)|0\rangle = 0, \quad (21)$$

$$T^{\text{tr}}(x)|0\rangle = 0. \quad (22)$$

To show the first two relations it is sufficient to show that χ annihilate the ground state. For that, we need to solve the equation (11) and find χ . First we act the operator \bar{D} on Eq. (11) and use Eq. (10) to get

$$\bar{D}D\chi + \bar{D}\mathcal{F} = 0, \quad \mathcal{F} = A_t\psi + \frac{\delta L_{\text{int}}}{\delta \psi^\dagger}. \quad (23)$$

Now we note that we can express the projection of any function $\mathcal{F}(\vec{x})$ onto the LLL as an expansion over derivatives,

$$\mathcal{F}_L(\vec{x}) = \mathcal{P}_{\text{LLL}}\mathcal{F}(\vec{x}) = \sum_{n=0}^{\infty} \frac{1}{n!B^n} D^n \bar{D}^n \mathcal{F}(\vec{x}). \quad (24)$$

In particular, one can check that this is consistent with

$$\bar{D}\mathcal{F}_L(\vec{x}) = 0, \quad (\mathcal{P}_{\text{LLL}})^2 = \mathcal{P}_{\text{LLL}}. \quad (25)$$

Replacing in Eq. (23) \mathcal{F} by $\mathcal{F} - \mathcal{F}_L$, then use the series expansion (24) for \mathcal{F}_{rmL} , one finally finds χ as a series

$$\chi = \sum_{n=1}^{\infty} \frac{1}{n!B^n} D^{n-1} \bar{D}^n \left(A_t\psi + \frac{\delta L_{\text{int}}}{\delta \psi^\dagger} \right). \quad (26)$$

In particular, if one considers noninteracting electrons, $L_{\text{int}} = 0$ and insert χ into the expression for the current, one reproduces the expression previously obtained by Martínez and Stone [10].

We now set $A_t = 0$ and inspect the operator χ . In both the bosonic and the fermionic cases, taking the variation over ψ^\dagger leaves the strings of annihilation operators, ψ^k and $\psi D\psi D^2\psi \dots$ intact on the right of L_{int} . But the trial wavefunction is annihilated by this exact string of annihilation operators,

$$\psi^k(x)|0\rangle_{\text{bosonic}} = 0, \quad (27)$$

$$\psi D\psi D^2\psi \dots D^{k-1}\psi|0\rangle_{\text{fermionic}} = 0. \quad (28)$$

We thus conclude that $\chi(x)$ annihilates the ground state, and from the explicit forms for the current and stress tensor, one concludes that the antiholomorphic components \bar{J} and \bar{T} annihilates the ground state.

Similar calculations also show that the trace of the stress tensor T^{tr} also annihilates the ground state.

The properties (20) are properties specific for the trial ground states and the Hamiltonian for which these ground states are exact zero-energy states. They are not valid for generic interaction, for example the Coulomb interactions.

Ward identities.—In flat spacetime we have the conservation laws for the particle number and momentum, which is in our notation are

$$\partial_t \rho + \bar{\partial}J + \partial\bar{J} = 0, \quad (29)$$

$$\partial\tilde{T} + \bar{\partial}T^{\text{tr}} = -iB\bar{J}, \quad (30)$$

$$\partial T^{\text{tr}} + \bar{\partial}T = +iBJ. \quad (31)$$

The last two equations are simply the force balance equations, since we are working in the limit $m \rightarrow 0$ where there is no inertia. We now sandwich these equations between the ground state $|0\rangle$ and an arbitrarily chosen state $\langle n|$, assuming that the latter is a state with zero particle number and carries energy E_n and momentum \mathbf{P}_n . Since \bar{J} , \tilde{T} and T^{tr} annihilate the ground state, the Ward identities imply direct proportionality between the matrix elements of the operators ρ , J and T ,

$$\langle n|\rho|0\rangle = -\frac{(P_n^x + iP_n^y)^2}{BE_n} \langle n|T_{zz}|0\rangle, \quad (32)$$

$$\langle n|J_z|0\rangle = -\frac{P_n^x + iP_n^y}{B} \langle n|T_{zz}|0\rangle. \quad (33)$$

This means that the three operators ρ , J and T create the same set of states at nonzero momentum, only with different matrix elements. For example, if a magneto-roton [7] exists it can be created equally well by all three operators. This does not apply to states with zero momentum (including the magneto-roton if it exists at zero momentum); these states cannot be created by acting ρ or J on the ground state, but may be created by the operator T .

If one introduces the spectral densities of the density and the holomorphic component of the stress tensor,

$$S(\omega, k) = \frac{1}{N} \sum_n |\langle n|\rho(\mathbf{k})|0\rangle|^2 \delta(\omega - E_n), \quad (34)$$

$$\rho_T(\omega, k) = \frac{1}{N} \sum_n |\langle n|T_{zz}(\mathbf{k})|0\rangle|^2 \delta(\omega - E_n), \quad (35)$$

then

$$\omega^2 S(\omega, k) = \frac{k^4}{B^2} \rho_T(\omega, k). \quad (36)$$

The static structure factor can be expressed as

$$S(k) = \int_0^\infty d\omega S(\omega, k) = k^4 \int_0^\infty \frac{d\omega}{\omega^2} \rho_T(\omega, k). \quad (37)$$

So at $k \rightarrow 0$ the static structure factor $S(k)$ is proportional to k^4 . Since by definition our spectral density $S(\omega, k)$ does not include the cyclotron mode, $S(k)$ is actually the projected structure factor $\bar{s}(k)$, which is known to be $O(k^4)$ at small k .

On the other hand, the retarded Green function of two components of the stress tensor T^{ij} and T^{kl} can be decomposed as [15, 16]:

$$G_R^{T^{ij}, T^{kl}}(\omega, \vec{0}) = K(\omega) \mathcal{I}_B^{ijkl} + \mu(\omega) \mathcal{I}_S^{ijkl} - i\omega \eta_H(\omega) \mathcal{I}_H^{ijkl}, \quad (38)$$

where $K(\omega)$, $\mu(\omega)$ and $\eta_H(\omega)$ are the frequency-dependent bulk modulus, shear modulus and Hall viscosity, respectively, and

$$\begin{aligned} \mathcal{I}_B^{ijkl} &= \delta^{ij} \delta^{kl}, \\ \mathcal{I}_S^{ijkl} &= \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} - \delta^{ij} \delta^{kl}, \\ \mathcal{I}_H^{ijkl} &= \frac{1}{2} (\delta^{ik} \epsilon^{jl} + \delta^{il} \epsilon^{jk} + \delta^{jk} \epsilon^{il} + \delta^{jl} \epsilon^{ik}), \end{aligned}$$

For the three independent response functions on ground states with $\tilde{T}|0\rangle = 0$, the analytic structure of the retarded 2-point function implies the following sum rule,

$$\int_0^\infty \frac{d\omega}{\omega^2} \rho_T(\omega) = \frac{\eta_H(0) - \eta_H(\infty)}{2\rho}, \quad (39)$$

Compared with (37), we have for our ground state

$$\lim_{k \rightarrow 0} \frac{S(k)}{k^4} = \frac{\eta_H(0) - \eta_H(\infty)}{2\rho}, \quad (40)$$

Now let us recall that the Hall viscosity (at zero frequency) of a gapped quantum Hall state is equal to $\eta_H(0) = \rho\mathcal{S}/4$ where \mathcal{S} is the shift of the state [19]. At frequencies much higher than the energy scale set by the interaction, interactions do not play any roles and the Hall viscosity is given by the same formula as in the integer quantum Hall state of the lowest Landau level, $\eta_H(\infty) = \rho/4$. Therefore the previous equation can be written as

$$\lim_{k \rightarrow 0} \frac{S(k)}{k^4} = \frac{\mathcal{S} - 1}{8} \quad (41)$$

The fact that this relationship is valid for the Laughlin wavefunction is well known. What we have shown is that this relationship is valid for a much wider class of ground states. In fact, one can show that the relationship is valid whenever the ground state is annihilated by the

uniform component of the antiholomorphic component of the stress tensor,

$$\int d\mathbf{x} \tilde{T}(\mathbf{x})|0\rangle = 0 \quad (42)$$

The trial ground states and their corresponding interaction Hamiltonian satisfy a stronger constraint $\tilde{T}(x)|0\rangle = 0$.

Conclusion.—Our result, Eq. (1) shows that there is a special class of quantum Hall wavefunctions that saturate the inequality (2). In these wavefunctions, the leading k^4 behavior of the static structure factor is related to the shift. It is also clear that the relationship $s_4 = (\mathcal{S} - 1)/8$ cannot be valid for all gapped quantum Hall states. For examples, for states that have $\mathcal{S} < 1$, for example the $\nu = 2/3$ state ($\mathcal{S} = 0$) or the anti-Pfaffian state [17, 18] ($\mathcal{S} = -1$), $(\mathcal{S} - 1)/8 < 0$ while, due to the positivity of the dynamic structure factor $S(\omega, k)$, the coefficient s_4 has to be positive.

It is known that many of the trial wave functions have large overlaps with the true ground state of the Hamiltonian with Coulomb interaction. It is interesting to see if the numerical value of the k^4 coefficient in the structure factor is close to the value $(\mathcal{S} - 1)/8$ achieved by the trial wavefunctions.

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